数I演習問題 (１４，１５，１６組)
4月27日実施
テーマ：二変数関数の微分可能性, 連鎖律

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注意:A4 の紙一枚両面に収まるように書いてください.

(1) $f(x,y) = x^3 - 3xy + 3y^2 - 5x + 2$ が $(2,1,-3)$ で微分可能なことを微分可能の定義に戻って示し, そこでの接平面の式を求めよ. (微分可能の定義を再確認してもらう問題. 本当はすべての点で微分可能であるが具体的にするために敢えてこうした.)

(2) 次の関数を偏微分せよ.

$$(a)\ (u+v)^{\sin(uv)} \quad (b)\ \sin\{u\cos(u+v)\}$$

(どのように合成関数と思えるのか考えてください. それに気づけば多少計算が面倒ですがあとは慣れの問題です.)

(3) $f(x,y)$ は原点を除いたところで微分可能な関数であるとする. 極座標表示 $x = r\cos\theta$, $y = r\sin\theta$ を

$$\{(r,\theta) \mid 0 < r, 0 \le \theta < 2\pi\}$$

から $\mathbb{R}^2 \setminus \{0\}$ への写像と考える. このとき, f_x, f_y を f_r, f_θ で表せ. (連鎖律から出るのは f_r, f_θ を f_x, f_y で表す式. どうすれば逆に出来るかを考える. 問題の仮定は逆戻りが出来るために付けた条件だがあまり気にしなくてよい.)

(4) $f(x,y)$ は微分可能な関数であるとする. もし $f_x = f_y$ が恒等的に成り立つならば, $f(x,y) = g(x+y)$ と書けることを示せ. ($s = x + y$ ともうひとつうまく変数を決めることで連鎖律に持ち込む. $f(x,y)$ が微分可能という仮定は連鎖律が適用できるように付けた仮定. あまり気にしなくてよい.)

(5) 授業に対する要望があったらいつでも書いてください.

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(1)

$$\frac{\partial f}{\partial x} = 3x^2 - 3y - 5, \quad \frac{\partial f}{\partial y} = 6y - 3x$$

なので、

$$\frac{\partial f}{\partial x}(2,1) = 4 \qquad \frac{\partial f}{\partial y}(2,1) = 0$$

$$f(2,1) = -3$$

よって、

$$f(x,y) - \frac{\partial f}{\partial x}(2,1)(x-2) - \frac{\partial f}{\partial y}(2,1)(y-1) - f(2,1)$$

$$= x^3 - 3xy + 3y^2 - 5x + 2 - 4(x-2) + 3$$

$$= x^3 - 3xy + 3y^2 - 9x + 13$$

さて、$f(x,y)$ が $(2,1,-3)$ で全微分可能であることを示すには、

$$\lim_{(x,y)\to(2,1)} \frac{f(x,y) - \frac{\partial f}{\partial x}(2,1)(x-2) - \frac{\partial f}{\partial y}(2,1)(y-1) - f(2,1)}{\sqrt{(x-2)^2 + (y-1)^2}} = 0 \quad \cdots ①$$

を示せばよい。極限をとる式を $g(x,y)$ とおく。

$$\begin{cases} x-2 = r\cos\theta, \\ y-1 = r\sin\theta \end{cases}$$ とすると、

$(x,y)\to(2,1)$ のとき $r \to 0$

$g(x,y)$ において、

(分母) $= r$

であり、

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(分子) $= x^3 - 3xy + 3y^2 - 9x + 13$

$= (r\cos\theta+2)^3 - 3(r\cos\theta+2)(r\sin\theta+1) + 3(r\sin\theta+1)^2$
$\quad - 9(r\cos\theta+2) + 13$

$= r^3\cos^3\theta + r^2(6\cos^2\theta - 3\cos\theta\sin\theta + 3\sin^2\theta)$
$\quad + r(12\cos\theta - 3\cos\theta - 6\sin\theta + 6\sin\theta - 9\cos\theta)$
$\quad + (8 - 6 + 3 - 18 + 13)$

$= r^3\cos^3\theta + r^2(6\cos^2\theta - 3\cos\theta\sin\theta + 3\sin^2\theta)$

よって

$$g(x,y) = \frac{1}{r}\left\{r^3\cos^3\theta + r^2(6\cos^2\theta - 3\cos\theta\sin\theta + 3\sin^2\theta\right\}$$
$$= r(r\cos^3\theta + 6\cos^2\theta - 3\cos\theta\sin\theta + 3\sin^2\theta)$$

なので、

$(x,y) \to (2,1)$, つまり $r \to 0$ のとき
$g(x,y) \to 0$

よって ① が成立するので、$f(x,y)$ は $(2,1)$ で全微分可能である。

(2)

(a)

$f(u,v) = (u+v)^{\sin uv}$ とする。

$$\begin{cases} x = u+v \\ y = \sin uv \end{cases}$$ とすると

$f(u,v) = x^y$ であり、

$\dfrac{\partial f}{\partial x} = y x^{y-1}$ $\qquad \dfrac{\partial f}{\partial y} = x^y \log x$

$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial v} = 1 \quad \frac{\partial y}{\partial u} = v\cos uv, \quad \frac{\partial y}{\partial v} = u\cos uv$

よって

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= y x^{y-1}\cdot 1 + x^y \log x \cdot v\cos uv$$

$$= x^y \left(\frac{y}{x} + v\cos uv \log x\right)$$

$$= (u+v)^{\sin uv}\left(\frac{\sin uv}{u+v} + v\cos uv \log (u+v)\right)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= y x^{y-1}\cdot 1 + x^y \log x \cdot u\cos uv$$

$$= x^y \left(\frac{y}{x} + u\cos uv \log x\right)$$

$$= (u+v)^{\sin uv}\left(\frac{\sin uv}{u+v} + u\cos uv \log (u+v)\right).$$

(b)

$f(u, v) = \sin\{u\cos(u+v)\}$ とする.

$$\begin{cases} x = u \\ y = \cos(u+v) \end{cases}$$ とすると

$f(u, v) = \sin xy$ なので,

$\frac{\partial f}{\partial x} = y\cos xy \quad \frac{\partial f}{\partial y} = x\cos xy$

また,

$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial v} = 0 \quad \frac{\partial x}{\partial u} = -\sin(u+v) \quad \frac{\partial x}{\partial v} = -\sin(u+v)$

よって

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= y\cos xy \cdot 1 + x\cos xy\,\left(-\sin(u+v)\right)$$

$$= \left(y - x\sin(u+v)\right)\cos xy$$

$$= \left(\cos(u+v) - u\sin(u+v)\right)\cos\left\{u\cos(u+v)\right\}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= y\cos xy \cdot 0 + x\cos xy\cdot\left(-\sin(u+v)\right)$$

$$= -x\cos xy\,\sin(u+v)$$

$$= -u\sin(u+v)\cos\left\{u\cos(u+v)\right\}$$

(3)

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial r} = r\cos\theta$$

よって

$$f_r = f_x\cdot\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\cdot\sin\theta$$

$$f_\theta = f_x\cdot\frac{\partial x}{\partial \theta} + f_y\cdot\frac{\partial y}{\partial \theta} = f_x\cdot(-r\sin\theta) + f_y\cdot r\cos\theta$$

∴

$$\begin{pmatrix} f_r \\ f_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$J = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$ とすると

$$J^{-1} = \frac{1}{r(\cos^2\theta + \sin^2\theta)} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}$$ $\left((x, y) \neq (0, 0) \text{ より}, \ r \neq 0 \right)$

ゆえに

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix}$$

$$\therefore \begin{cases} f_x = f_r \cos\theta - f_\theta \cdot \dfrac{\sin\theta}{r} \\ f_y = f_r \sin\theta + \dfrac{\cos\theta}{r} f_\theta \end{cases}$$

(別解)

$x^2 + y^2 = r^2$ の両辺を x, y で偏微分して

$$2x = 2r \cdot \frac{\partial r}{\partial x} \qquad 2y = 2r \cdot \frac{\partial r}{\partial y}$$

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$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \quad \cdots\cdots ①$$

$x = r\cos\theta$ の両辺を x, y で偏微分して

$$\begin{cases} 1 = \dfrac{\partial r}{\partial x}\cos\theta - \dfrac{\partial \theta}{\partial x}\cdot r\sin\theta = \cos^2\theta - \dfrac{\partial\theta}{\partial x} r\sin\theta \\ 0 = \dfrac{\partial r}{\partial y}\cos\theta - \dfrac{\partial\theta}{\partial y}\cdot r\sin\theta = \cos\theta\sin\theta - \dfrac{\partial\theta}{\partial y} r\sin\theta \end{cases}$$

よって

$$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r} \qquad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r} \quad \cdots\cdots ②$$

①, ② より

$$f_x = f_r\frac{\partial r}{\partial x} + f_\theta\frac{\partial\theta}{\partial x} = f_r\cos\theta - f_\theta\cdot\frac{\sin\theta}{r}$$

$$f_y = f_r\frac{\partial r}{\partial y} + f_\theta\cdot\frac{\partial\theta}{\partial y} = f_r\sin\theta - f_\theta\cdot\frac{\cos\theta}{r}$$

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(4)

$$\begin{cases} s = x + y \\ t = x - y \end{cases}$$ とすると

$$\frac{\partial s}{\partial x} = 1 \quad \frac{\partial s}{\partial y} = 1 \quad \frac{\partial t}{\partial x} = 1 \quad \frac{\partial t}{\partial y} = -1$$

よって

$$f_x = f_s\frac{\partial s}{\partial x} + f_t\frac{\partial t}{\partial x} = f_s + f_t$$

$$f_y = f_s\frac{\partial s}{\partial y} + f_t\frac{\partial t}{\partial y} = f_s - f_t$$

よって $f_t = \dfrac{1}{2}(f_x - f_y) = 0 \quad (\because f_x = f_y)$

よって

$f(x, y)$ は t に依存せず, s のみの関数なので,

$f(x, y) = g(s) = g(x+y)$ の形で書ける。

(注).
別に
$t = x - y$ でなくても O.K. です。
実際
$t = g(x, y)$ とおくと

$$\begin{cases} f_x = f_s + g_x f_t \\ f_y = f_s + g_y f_t \end{cases}$$ なので、

$$(g_x - g_y) f_t = f_x - f_y = 0$$

だから、 $g_x - g_y \neq 0$ のとき
$f_t = 0$ が出ます。
ただ、 $g(x, y)$ を複雑にすると微分するのが面倒なうえ、
$g_x - g_y = 0$ となる場合も出てきます。
なので、
$t = x - y$ とか $t = x$ とか $t = y$
のような簡単な関数にしたほうがラクです。

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(1)

$f(x, y) = x^3 + y^3 - x - y$
は $(2, 1, 6)$ で全微分可能なことを示せ，接平面も求めよ。

(2)

$f(x, y) = \sin(x + y)$ は，各点 (a, b) で全微分可能なことを示せ。接平面も求めよ。

2　次の関数を，連鎖律を利用して偏微分せよ。

(1)

$$f(x, y) = (xy)^{e^{x+y}}$$

(2)

$$f(x, y) = \sin\{y \log(x+y)\}$$

(3)

$$f(x, y) = \log\left(\sin e^{x+y} + \cos e^{x-y}\right)$$

3

$f(x, y)$ は微分可能とする

$$\begin{cases} x = se^{t} \\ y = te^{s} \end{cases}$$

とするとき，

$\dfrac{\partial f}{\partial x}$，$\dfrac{\partial f}{\partial y}$ を $\dfrac{\partial f}{\partial s}$，$\dfrac{\partial f}{\partial t}$ であらわせ。

ただし $st \neq 1$ とする。

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4 何回でも（$(x, y) \neq (0, 0)$ で）微分可能な関数 $f(x, y)$ について，

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

なる極座標変換をする。このとき，

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$ を $\dfrac{\partial^2 f}{\partial r^2}, \dfrac{\partial^2 f}{\partial \theta^2}, \dfrac{\partial f}{\partial r}, \dfrac{\partial f}{\partial \theta}$

であらわせ。

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(1)

$\dfrac{\partial f}{\partial x} = 3x^2 - 1 \quad \dfrac{\partial f}{\partial y} = 3y^2 - 1$ より

$\dfrac{\partial f}{\partial x}(2,1) = 11 \quad \dfrac{\partial f}{\partial y}(2,1) = 2 \quad f(2,1) = 6$

よって

$$\lim_{(x,y)\to(2,1)} \frac{f(x,y) - \frac{\partial f}{\partial x}(2,1)(x-2) - \frac{\partial f}{\partial y}(2,1)(y-1) - f(2,1)}{\sqrt{(x-2)^2 + (y-1)^2}} = 0 \quad \cdots\cdots ①$$

を示せばよい。①で極限をとる式を $g(x,y)$ とおく。

$$\begin{cases} x-2 = r\cos\theta \\ y-1 = r\sin\theta \end{cases}$$ とおくと

($g(x,y)$ の分母) $= r$

($g(x,y)$ の分子)

$= x^3 + y^3 - x - y - 11(x-2) - 2(y-1) - 6$

$= x^3 + y^3 - 12x - 3y + 18$

$= (r\cos\theta + 2)^3 + (r\sin\theta + 1)^3 - 12(r\cos\theta + 2)$
$\quad - 3(r\sin\theta + 1) + 18$

$= r^3(\cos^3\theta + \sin^3\theta) + r^2(6\cos^2\theta + 3\sin^2\theta)$
$\quad + r(12\cos\theta + 3\sin\theta - 12\cos\theta - 3\sin\theta)$
$\quad + (8 + 1 - 24 - 3 + 18)$

$= r^3(\cos^3\theta + \sin^3\theta) + r^2(6\cos^2\theta + 3\sin^2\theta)$

よって $g(x,y) = r(r\cos^3\theta + r\sin^3\theta + 6\cos^2\theta + 3\sin^2\theta)$

だから、$(x,y) \to (2,1)$，つまり $r \to 0$ のとき

$g(x,y) \to 0$ 　　よって $\displaystyle\lim_{(x,y)\to(2,1)} g(x,y) = 0$

なので、①は成立する。よって $g(x,y)$ は $(2,1,6)$ で全微分可能

また、接平面の式は

$$z = \frac{\partial f}{\partial x}(2,1)(x-2) + \frac{\partial f}{\partial y}(2,1)(y-1) + f(2,1)$$

$$= 11(x-2) + 2(y-1) + 6$$

$$= 11x + 2y - 18$$

$$\therefore \quad z = 11x + 2y - 18$$

(2)

$$\frac{\partial f}{\partial x} = \cos(x+y) \qquad \frac{\partial f}{\partial y} = \cos(x+y)$$

なので、$(a, b, \sin(a+b))$ において、

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = \cos(a+b)$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - \frac{\partial f}{\partial x}(x-a) - \frac{\partial f}{\partial y}(y-b) - f(a,b)}{\sqrt{(x-a)^2+(y-b)^2}} = 0 \quad \cdots ①$$

を示せばいい。極限をとる式を $g(x,y)$ とする。

$$\begin{cases} x-a = r\cos\theta \\ y-b = r\sin\theta \end{cases}$$

とおくと

$(g(x,y)$ の分母$) = r$

$(g(x,y)$ の分子$)$

$$= \sin(x+y) - \cos(a+b)(x-a) - \cos(a+b)(y-b) - \sin(a+b)$$

$$= \sin(x+y) - 2r\cos(a+b) - \sin(a+b)$$

よって

$$g(x,y) = \frac{\sin(x+y)-\sin(a+b)}{r} - \cos(a+b)$$

$$= \frac{2}{r}\sin\frac{(x-a)+(y-b)}{2}\cos\frac{x+y+a+b}{2} - 2\cos\theta$$

$$= \frac{2\sin r}{r}\cos\left(a+b+\frac{r\cos\theta+r\sin\theta}{2}\right) - 2\cos\theta$$

よって、

$(x,y)\to(a,b)$ のとき、 $r\to 0$ より、

$$g(x,y) \to 2\cdot 1\cdot\cos(a+b+0) - 2\cos(a+b) = 0$$

よって①は成立するので、$f(x,y)$ は (a,b) で全微分可能で、接平面は、

$$z = \cos(a+b)(x-a) + \cos(a+b)(y-b) + \sin(a+b)$$

$$= (x+y-a-b)\cos(a+b) + \sin(a+b)$$

$$\therefore\ z = (x+y-a-b)\cos(a+b) + \sin(a+b)$$

2

(1) $\begin{cases} u = xy \\ v = e^{x+y} \end{cases}$ とおいて

$$f(x,y) = u^v$$

$$\frac{\partial f}{\partial u} = vu^{v-1} \qquad \frac{\partial f}{\partial v} = u^v \log u$$

$$\frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x \quad \frac{\partial v}{\partial x} = e^{x+y} \quad \frac{\partial v}{\partial y} = e^{x+y}$$

よって

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= vu^{v-1}\cdot y + u^v\log u\cdot e^{x+y}$$

$$= u^{v}\left(\frac{yv}{u} + e^{x+y}\log u\right)$$

$$= (xy)^{e^{x+y}}\left(\frac{e^{x+y}}{x} + e^{x+y}\log(x+y)\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= vu^{v-1}\cdot x + u^{v}\log u\cdot e^{x+y}$$

$$= u^{v}\left(\frac{xv}{u} + e^{x+y}\log u\right)$$

$$= (xy)^{e^{x+y}}\left(\frac{e^{x+y}}{y} \quad e^{x+y}\log(x+y)\right)$$

(2)

$u = y$
$v = \log(x+y)$ とおくと

$f(x, y) = \sin uv$

$$\frac{\partial f}{\partial u} = v\cos uv \qquad \frac{\partial f}{\partial v} = u\cos uv$$

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = \frac{1}{x+y} \qquad \frac{\partial v}{\partial y} = \frac{1}{x+y}$$

よって

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= u\cos uv\cdot\frac{1}{x+y} = \frac{y}{x+y}\cos\{y\log(x+y)\}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= v\cos uv + u\cos uv\cdot\frac{1}{x+y}$$

$$= \cos\{y\log(x+y)\}\log(x+y) + \frac{y}{x+y}\cos\{y\log(x+y)\}$$

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(3)

$$\begin{cases} u = e^{x+y} \\ v = e^{x-y} \end{cases}$$ とおくて

$$f(x, y) = \log(\sin u + \cos v)$$

$$\frac{\partial f}{\partial u} = \frac{\cos u}{\sin u + \cos v} \quad \frac{\partial f}{\partial v} = \frac{-\sin v}{\sin u + \cos v}$$

$$\frac{\partial u}{\partial x} = e^{x+y} \quad \frac{\partial u}{\partial y} = e^{x+y} \quad \frac{\partial v}{\partial x} = e^{x-y} \quad \frac{\partial v}{\partial y} = -e^{x-y}$$

よって

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{e^{x+y}\cos u - e^{x-y}\sin v}{\sin u + \cos v}$$

$$= \frac{e^{x+y}\cos e^{x+y} - e^{x-y}\sin e^{x-y}}{\sin e^{x+y} + \cos e^{x-y}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{e^{x+y}\cos u + e^{x-y}\sin v}{\sin u + \cos v}$$

$$= \frac{e^{x+y}\cos e^{x+y} + e^{x-y}\sin e^{x-y}}{\sin e^{x+y} + \cos e^{x-y}}$$

3

$$\frac{\partial x}{\partial s} = e^t \quad \frac{\partial x}{\partial t} = se^t \quad \frac{\partial y}{\partial s} = te^s \quad \frac{\partial y}{\partial t} = e^s$$

よって

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = e^t\frac{\partial f}{\partial x} + te^s\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = se^t\frac{\partial f}{\partial x} + e^s\frac{\partial f}{\partial y}$$

$$\begin{pmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial s} \end{pmatrix} = \begin{pmatrix} e^t & te^s \\ se^t & e^s \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\therefore \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \frac{1}{(1-st)e^{s+t}} \begin{pmatrix} e^s & -te^s \\ -se^t & e^t \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial s} \end{pmatrix}$$

$$\therefore \begin{cases} \frac{\partial f}{\partial x} = \frac{1}{(1-st)e^t} \frac{\partial f}{\partial x} - \frac{t}{(1-st)e^t} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial y} = -\frac{s}{(1-st)e^s} \frac{\partial f}{\partial x} + \frac{1}{(1-st)e^s} \frac{\partial f}{\partial y} \end{cases}$$

4

$$\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

よって

$$\begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot(-r\sin\theta) + \frac{\partial f}{\partial y}\cdot r\cos\theta \end{cases}$$

$$\therefore \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\therefore \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{pmatrix}$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \cos\theta \cdot \dfrac{\partial f}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial \theta} & \text{---} ① \\ \dfrac{\partial f}{\partial y} = \sin\theta \cdot \dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial \theta} & \text{---} ② \end{cases}$$

①より

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\cdot\frac{\sin\theta}{r}\right)$$

$$= \cos\theta \cdot \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\cdot\frac{\sin\theta}{r}\right)$$

$$- \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\cdot\frac{\sin\theta}{r}\right)$$

$$= \cos\theta\left(\frac{\partial^2 f}{\partial r^2}\cos\theta - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} + \frac{\sin\theta}{r^2}\frac{\partial f}{\partial \theta}\right)$$

$$- \frac{\sin\theta}{r}\left(\frac{\partial^2 f}{\partial\theta\partial r}\cos\theta - \sin\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial \theta^2} - \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{\partial^2 f}{\partial r^2}\cos^2\theta + \frac{\partial^2 f}{\partial \theta^2}\cdot\frac{\sin^2\theta}{r^2} - \frac{\partial^2 f}{\partial\theta\partial r}\cdot\frac{2\sin\theta\cos\theta}{r}$$

$$+ \frac{\partial f}{\partial r}\cdot\frac{\sin^2\theta}{r^2} + \frac{\partial f}{\partial \theta}\cdot\frac{2\sin\theta\cos\theta}{r^2} \quad \text{---} ③$$

②より

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$+ \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$= \sin\theta\left(\sin\theta\frac{\partial^2 f}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r} - \frac{\cos\theta}{r^2}\frac{\partial f}{\partial \theta}\right)$$

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$$+\frac{\cos\theta}{r}\left(\frac{\partial^2 f}{\partial\theta\partial r}\sin\theta+\frac{\partial f}{\partial r}\cos\theta+\frac{\partial^2 f}{\partial\theta^2}\cdot\frac{\cos\theta}{r}-\frac{\partial f}{\partial\theta}\cdot\frac{\sin\theta}{r}\right)$$

$$=\frac{\partial^2 f}{\partial r^2}\sin^2\theta+\frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial\theta\partial r}\cdot\frac{2\sin\theta\cos\theta}{r}$$

$$+\frac{\partial f}{\partial r}\cdot\frac{\cos^2\theta}{r}-\frac{\partial f}{\partial\theta}\cdot\frac{2\sin\theta\cos\theta}{r^2} \quad \cdots\cdots (4)$$

(3) + (4) により

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r}\frac{\partial f}{\partial r}.$$